Note, in conclusion, that the geometric method of constructing the FE basis, distinguished by simplicity and universality, is especially effective in modeling elements of higher orders of approximation.

NOTATION

T, temperature; T_i, nodal temperature values; Φ_i, basis functions of the finite element; ρ, θ, ζ, cylindrical coordinates.

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SOLVING NONSTEADY HEAT-CONDUCTION PROBLEMS FOR MULTILAYER SYSTEMS BY THE FINITE-DIFFERENCE METHOD

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The problem of heat propagation in a multilayer system with intrinsic heat liberation of any subsystem depending on the temperature, coordinates, and time is considered.

The construction of concrete housings of underground structures in various climatic conditions requires optimization of the temperature conditions of concreting. In practice, temperature regulation is accomplished either by producing definite conditions of concrete heating in the jacket (thermal-heating method) or by heating the housing by means of insulational materials in contact with air (thermos method). Theoretical analysis of the choice of parameters of the optimal conditions reduces to solving the problem of nonsteady heat conduction in a multilayer system.

Suppose that, within the limits of each subsystem, the thermophysical characteristics are constants and the heat liberation of any subsystem may be represented by a specified function of the temperature, time, and coordinates Q = Q(U, r, t). Then heat propagation in the system may be described by the following nonlinear equation of nonsteady heat conduction for the one-dimensional case in cylindrical coordinates

$$\frac{r}{c_v}\frac{\partial Q}{\partial t} = r\frac{\partial U}{\partial t} - \frac{\partial}{\partial r}\left(ar\frac{\partial U}{\partial r}\right), \quad r_{\mathrm{L}} < r < r_{\mathrm{re}} \tag{1}$$

with initial condition

$$U(r,\ 0) = \varphi(r), \quad r_{\mathrm{L}} < r < r_{\mathrm{re}} \tag{2}$$

and boundary conditions of the first kind

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$$U(r_L, t) = \gamma^0(t), \quad U(r_{re}, t) = \gamma^1(t), \quad 0 < t < T \tag{3}$$

for the first boundary problem or of the second and third kind

$$\begin{aligned}(-ar_L)\partial U(r_L, t)/\partial r + \beta^0(t)U(r_L, t) = \gamma^0(t), \quad 0 < t < T,\\ (ar_{re})\partial U(r_{re}, t)/\partial r + \beta^1(t)U(r_{re}, t) = \gamma^1(t), \quad 0 < t < T\end{aligned} \tag{4}$$

for the second and third boundary problems.

In cases that are of practical importance, approximate methods of solving Eq. (1) are used — in particular, the finite-difference method [1]. The best-known method of obtaining calculational difference formulas for the temperature fields is the method based on physical concepts regarding the thermal balance of elementary volumes. In solving Eq. (1) by a finite-difference method, the calculational points may be within the limits of a single system (boundary conditions of the first and second kind) or may lie on the boundary with air (boundary conditions of the third kind) or may be on the boundary of two inhomogeneous media (boundary conditions of the fourth kind). Within the framework of the thermal-balance model for boundary conditions of the first, second, and third kind in cylindrical coordinates, calculational formulas written with an accuracy up to terms in $(\Delta r)^2$ are known. With boundary conditions of the fourth kind, no such formulas are known.

It is expedient to obtain calculational formulas using the Marchuk integral identities [2, 3]. With their use, in the present work, it is relatively simple to find theoretical formulas for boundary conditions of the first, second, and third kinds coinciding with the analogous formulas obtained on the basis of the thermal balance, and a formula for the boundary condition of the fourth kind is derived for the first time for media with intrinsic heat liberation depending on the temperature in the case of a cylindrical thermal front. First, for a single medium, the linear heat-transfer equation of the type in Eq. (1) is considered, where the heat liberation Q is a function of the time and coordinates, and the results are generalized to the case in which the heat liberation also depends on the temperature.

The region of variation of r and t is covered by a grid formed by the points r_k, t_l: $r_k = k\Delta r$, $t_l = l\Delta t$ ($k = 0, 1, \ldots, n$; $l = 0, 1, \ldots, m$; $\Delta r = h/n$; $\Delta t = T/m$), where h is the spatial dimension of the region and T is the characteristic time. Then the system of difference equations for internal points of the region is derived using the first Marchuk integral identity

$$\int\limits_{t_{l-1}}^{t_l} \frac{U(r_{k+1}, t) - U(r_k, t)}{\int\limits_{r_k}^{r_{k+1}} \frac{dr}{r}} dt - \int\limits_{t_{l-1}}^{t_l} \frac{U(r_k, t) - U(r_{k-1}, t)}{\int\limits_{r_{k-1}}^{r_k} \frac{dr}{r}} dt = \int\limits_{t_{l-1}}^{t_l} dt \int\limits_{r_{k-\frac{1}{2}}}^{r_{k+\frac{1}{2}}} \left(\frac{s}{a} \frac{\partial U(s, t)}{\partial t} - \frac{s}{ac_v} \frac{\partial Q(s, t)}{\partial t} \right) ds,$$

where $U(r_{k+1})$, $U(r_k)$, $U(r_{k-1})$ are the temperature values at adjacent calculation points at time t. Integration in Eq. (5) gives

$$\begin{aligned}U(r_k, t_l) = U(r_k, t_{l-1}) \left[1 + \frac{a\Delta t}{(\Delta r)^2} \frac{\Delta r}{r_k \ln\left(1 - \frac{\Delta r}{r_k}\right)} - \frac{a\Delta t}{(\Delta r)^2} \frac{\Delta r}{r_k \ln\left(1 + \frac{\Delta r}{r_k}\right)} \right] - \\ - U(r_{k-1}, t_{l-1}) \frac{a\Delta t}{(\Delta r)^2} \frac{\Delta r}{r_k \ln\left(1 - \frac{\Delta r}{r_k}\right)} + U(r_{k+1}, t_{l-1}) \frac{a\Delta t}{r_k \ln\left(1 + \frac{\Delta r}{r_k}\right)} + \frac{Q(r_k, t_l) - Q(r_k, t_{l-1})}{c_v}.\end{aligned} \tag{6}$$

The second Marchuk identity is written for the derivative of the temperature with respect to the coordinate $\partial U(r, t)/\partial r$. At the left-hand boundary of the region of definition with respect to r

$$\int\limits_{t_{l-1}}^{t_l} r_L \frac{\partial U(r_L, t)}{\partial r} dt = \int\limits_{t_{l-1}}^{t_l} \frac{dt}{\int\limits_{r_L}^{r_L+\Delta r} \frac{dr}{r}} [U(r_L + \Delta r, t) - U(r_L, t)] \tag{7}$$

$$-\int_{t_{l-1}}^{t_l} \frac{dt}{\int_{r_L}^{r_L+\Delta r} \frac{dr}{r}} \int_{r_L}^{r_L+\Delta r} \frac{dr}{r} \int_{r_L}^{r} \left(\frac{s}{a} \frac{\partial U(s,\ t)}{\partial t} - \frac{s}{ac_v} \frac{\partial Q(s,\ t)}{\partial t} \right) ds.$$

The integral identity for $\partial U/\partial r$ at the right-hand edge of the region of definition with respect to r is written analogously.

Using Eq. (7), a difference equation is derived for the boundary condition of the third kind corresponding to the boundary condition in Eq. (4) when $\beta^0(t) \neq 0$ and $\beta^1(t) \neq 0$. The boundary condition at boundary r_L at time t is rewritten, taking $\beta^0(t) = r_L\alpha/c_V$, $\gamma^0(t) = r_L\alpha t^*/c_V$

$$\alpha(U(r_L\ t) - t^*) = ac_v\partial U(r_L\ t)/\partial r = \lambda\partial U(r_L\ t)/\partial r, \tag{8}$$

where t^* is the temperature of the surrounding air. Integrating Eq. (8) over time on the segment $[t_{l-1},\ t_l]$, taking account of the Marchuk identity in Eq. (7), a finite-difference equation of medium—air or medium—liquid heat transfer is obtained

$$U(r_L\ t_l) = U(r_L,\ t_{l-1}) \left[1 - \frac{2a\Delta t}{\ln\left(1+\frac{\Delta r}{r_L}\right) r_L\Delta r\left(1+\frac{\Delta r}{6r_L}\right)} - \frac{2a\alpha\Delta t}{\lambda\left(1+\frac{\Delta r}{6r_L}\right)}\right] + t^* \frac{2a\alpha\Delta t}{\lambda\Delta r\left(1+\frac{\Delta r}{6r_L}\right)} +$$
$$U(r_L+\Delta r,\ t_{l-1}) \frac{2a\Delta t}{\ln\left(1+\frac{\Delta r}{r_L}\right) r_L\Delta r\left(1+\frac{\Delta r}{6r_L}\right)} + \frac{Q(r_L\ t_l) - Q(r_L\ t_{l-1})}{c_v}. \tag{9}$$

Since the calculation formulas for the temperature have the explicit form of difference equations, the following constraints must be imposed on Δr and Δt for convergence of the approximate formulas [1]

$$\Delta t \leqslant (\Delta r)^2/2a \tag{10}$$

for the first and second boundary problems and

$$\Delta t \leqslant \frac{(\Delta r)^2}{2a\left(1+\frac{\alpha\Delta r}{\lambda}\right)} \frac{r_L}{(r_L + h)} \tag{11}$$

for the third [3].

Using Eq. (7), difference formulas are obtained for the contact of two inhomogeneous media (boundary condition of the fourth kind). Without loss in generality, it is assumed that to the left of the boundary there is a nonthermoactive medium with the thermophysical characteristics a_1, λ_1, α_1 and to the right there is a thermoactive medium with the thermophysical characteristics a_2, λ_2, α_2. With ideal contact, the usual matching conditions must be satisfied

$$\begin{gathered} U(r_b\ t_l)\big|_{r_b-0} = U(r_b,\ t_l)\big|_{r_b+0}, \\ \lambda_1\left(\frac{\partial U(r_b\ t_l)}{\partial r}\right)\bigg|_{r_b-0} = \lambda_2\left(\frac{\partial U(r_b\ t_l)}{\partial r}\right)\bigg|_{r_b+0}. \end{gathered} \tag{12}$$

The condition of equal heat-flux densities to the left and right of contact must be satisfied at any moment of time. Hence, the corresponding time integrals over the interval Δt must also be equal

$$\lambda_1 \int_{t_{l-1}}^{t_l} \frac{\partial U(r_b\ t)}{\partial r}\bigg|_{r_b-0} dt = \lambda_2 \int_{t_{l-1}}^{t_l} \frac{\partial U(r_b\ t)}{\partial r}\bigg|_{r_b+0} dt. \tag{12a}$$

Estimating the derivatives in Eq. (12a) using Eq. (7), a difference expression is obtained for the temperature at the contact of two inhomogeneous media, taking account of the intrinsic heat liberation

$$U(r_l,\ t_l)=U(r_b,t_{l-1})\left[1-\frac{1}{(v_1c_1+v_2c_2)}\left(\frac{\Delta t}{R_1}+\frac{\Delta t}{R_2}\right)\right]+U(r_b-\Delta r_1,\ t_{l-1})\frac{\Delta t}{(v_1c_1+v_2c_2)R_1}+U(r_b+\Delta r_2,\ t_{l-1})\times$$
$$\times\frac{\Delta t}{(v_1c_1+v_2c_2)R_2}+\frac{c_2v_2}{(c_1v_1+c_2v_2)}\,\frac{Q(r_b,t_l)-Q(r_b,\ t_{l-1})}{c_2}. \quad (13)$$

The following notation is introduced in Eq. (13):

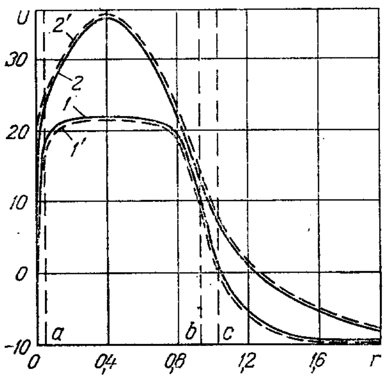

Fig. 1. Dependence of the temperature U (°C) in a multilayer system (heat insulation—concrete—heat insulation—rock) on the distance r, m: a, b, c, boundaries; 1, 1') temperatures at a time 10 h after laying concrete at $\Delta\tau_1$ = 0.08 h and $\Delta\tau_2$ = 0.02 h, respectively; 2, 2') the same, but after 50 h.

$$\frac{\lambda_1}{a_1}=c_1,\quad -\frac{\ln\left(1-\frac{\Delta r_1}{r}\right)}{\lambda_1}=R_1,\quad \frac{\Delta r_1 r_b}{2}\left(1-\frac{\Delta r_1}{6r_b}\right)=v_1,$$
$$\frac{\lambda_2}{a_2}=c_2,\quad \frac{\ln\left(1+\frac{\Delta r_2}{r_b}\right)}{\lambda_2}=R_2,\quad \frac{\Delta r_2 r_b}{2}\left(1+\frac{\Delta r_2}{6r_b}\right)=v_2.$$

To find the stability conditions for a multilayer system, equations of the type in Eqs. (10) and (11) must be written for each subsystem. The spatial intervals Δr_1, Δr_2, ..., Δr_n may be chosen in the form of the corresponding intervals from the given inequalities, and the unit time interval may be taken to be the minimal value of Δt_1, Δt_2, ..., Δt_n. In this case, the difference formulas will also be stable for a boundary condition of the fourth kind [1, 3].

The solutions obtained are now generalized to the nonlinear case, when the heat liberation Q depends on the temperature — Q = Q(U, r, t) — by the method of successive approximation. Suppose that at time t_l the solution of the nonlinear equation U(r, t) is known. To obtain the solution at time $t_l + \Delta t$, the known solution U(r, t) is substituted into the expression for $\partial Q(U, r, t)/\partial t$, turning the derivative $\partial Q/\partial t$ into a function of the coordinate and time $\partial Q(U, r, t)/\partial t = \partial\tilde{Q}(r, t)/\partial t$. Then, using the Marchuk identity, it is found that

$$U_{l+1}(r_k,\ t_l+\Delta t)=U_l(r_{k-1},\ t_l)K_{-1}+U_l(r_k,\ t_l)K_0+U_l(r_{k+1},\ t_l)K_{+1}+\Delta U(r_k,\ t_l+\Delta t), \quad (14)$$

where

$$\Delta U(r_k,\ t_l+\Delta t)=Q_{l+1}(U_l,\ r_k,\ t_{l+1}+\Delta t)/c_v-Q_l(U_{l-1},\ r_k,\ t_l)/c_v;\quad \Delta U\ll U. \quad (15)$$

The coefficients K_{-1}, K_0, K_{+1} coincide with the corresponding coefficients at temperatures $U(r_{k-1}, t_l)$, $U(r_k, t_l)$, $U(r_{k+1}, t_l)$ in Eqs. (6), (9), and (13). In accordance with the requirements of the method of successive approximation, the condition of smallness of the correction in comparison with the total solution must be tested at each time step. In order to confirm the convergence of the solutions obtained in the nonlinear case, it is necessary to take the sequence of time intervals $\Delta t_1 < \Delta t_2 < \ldots < \Delta t_n$ and compare the solutions at the corresponding time. If they coincide with specified accuracy, it may be supposed that the solution is convergent in the nonlinear equation.

The formulas obtained are used to calculate the temperature fields in the concrete housing of tunnels constructed in excavated rock by the thermos method. A thermal-insulation layer is in contact with the rock, the concrete is placed in a mold including a thermal-insulation coating. The whole system is represented in the form of a four-layer cylinder, each layer of which has constant thermophysical characteristics and dimensions. Thermal contact between the layers is assumed to be ideal; the air temperature in the tunnel is constant. Heat liberation of the concrete housing is taken into account by the formula of [4], which takes the following finite-difference form:

$$Q_{m+1} = Q_{\max}\left\{1 - \left[1 + \Delta t A_{20} \sum_{l=0}^{m} \exp\left[\ln 2 \frac{U(r_k, t_l) - 20}{8 + 0{,}13U(r_k, t_l)}\right]\right]^{-0{,}833}\right\}, \qquad (16)$$

where Q_{max}, A_{20} are reaction constants of the heat liberation of concrete characterizing the maximum possible heat liberation (J/m^3) and reaction rate (h^{-1}) determined experimentally, respectively. Calculations of the temperature distribution in the system have been performed on a computer using Eqs. (6), (9), and (13) for various sets of geometric and thermophysical characteristics of the layers, and also with different values of the time step Δt.

As an example, the results of numerical calculation of the temperature distribution in a four-layer system at $\Delta t_1 = 0.08$ h and $\Delta t_2 = 0.02$ h are shown in Fig. 1. The geometric and thermophysical characteristics are as follows: layer thickness $h_1 = 0.04$ m, $h_2 = 0.9$ m, $h_3 = 0.1$ m, $h_4 = 4$ m; the thermal diffusivity, thermal conductivity, and heat-transfer coefficient, in accordance with [5], are: $a_1 = 0.00046\ m^2/h$, $\lambda_1 = 0.174$ W/m·deg, $\alpha_1 = 7.5\ W/m^2\cdot deg$, $a_2 = 0.0027\ m^2/h$, $\lambda_2 = 1.45$ W/m·deg, $\alpha_2 = 7.5\ W/m^2\cdot deg$, $a_3 = 0.0027\ m^2/h$, $\lambda_3 = 1.45$ W/m·deg, $\alpha_3 = 15\ W/m^2\cdot deg$; $a_4 = 0.0042\ m^2/h$, $\lambda_4 = 3$ W/m·deg. The values in [6] are used for the reaction constants of heat liberation: $Q_{max} = 132\ kJ/m^3$, $A_{20} = 0.0138\ hr^{-1}$.

It is evident from Fig. 1 that, even with very different values of the variables Δt_1 and Δt_2, the curves of the calculational temperatures practically coincide. This indicates stability of the approximate solutions of the heat-conduction Eq. (1) in multilayer media with possible intrinsic heat liberation depending on the temperature.

NOTATION

c_V, specific heat, $kJ/m^3\cdot deg$; a, thermal diffusivity, m^2/h; α, heat-transfer coefficient of medium, $W/m^2\cdot deg$; λ, thermal conductivity, W/m·deg; U, temperature, K; r, coordinate, m; t, time, h; Δr, spatial interval, m; Δt, time interval, h; h, geometric dimension of system, m.

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